Topics on Particle Physics in the Early Universe

[Read as: QFT in de Sitter space]

Answer 1: Cosmology: Our universe $\rightarrow$ dS, $t \rightarrow \pm\infty$

$t \rightarrow -\infty$, cosmic inflation $H \sim 10^{14}$ GeV.

$t \rightarrow +\infty$, "dark energy"

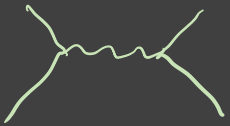
Answer 2: Particle physics: (LHC, 10^4 GeV) $\rightarrow$ "10^5 GeV in 50 yrs?

$M_{Pl} \simeq 2.4 \times 10^{18}$ GeV.

Answer 3: (Conceptual) dS $>$ Mink $>$ AdS

dS is arguably the most complicated space among all max. sym. manifolds in 4d.

Answer 4: (Technical) QFT (amplitudes) in dS is hard!



[ZZ Xianyu @ SEU Summer School 2021]

[A far too ambitious] Plan of the course

1. Cosmic Inflation

☐ Why inflation?
☐ Background dynamics
☐ Linear perturbation theory
☐ Observables
☐ Inflation models*

2. QFT in dS

☐ dS: geometries, symmetries
☐ Penrose diagram, causal properties
☐ Free fields and quantization
☐ BD vacuum is thermal
☐ Classification of states; UIRs

3. Cosmic correlators

☐ Schwinger-Keldysh path integral
☐ Perturbation theory to 3rd order
☐ Local non-Gaussianity
☐ Consistent relation*
☐ EFT of inflation and equilateral non-G*

4. Cosmological Collider Physics

☐ Soft limits as a discovery channel
☐ 4-point correlator in collapsed limit
☐ 3-point correlator
☐ Loop effects*
☐ Recent progress and prospects

References I find useful (especially those in Green)

Cosmology in general

☐ Dodelson: *Modern cosmology*
☐ Weinberg: *Cosmology*
☐ Mukhanov: *Physical Foundations of Cosmology*

Inflation and non-Gaussianity

☐ Baumann & McAllister 1404.2601
☐ Baumann: 0907.5424
☐ Maldacena: astro-ph/0210603
☐ Chen: 1002.1416
☐ Wang: 1303.1523

QFT in de Sitter space

☐ Spradlin, Strominger Volovich: hep-th/0110007
☐ Akhmedov: 1309.2557

Schwinger-Keldysh formalism

☐ Jordan: PRD 33 (1986) 444
☐ Chen, Wang, ZZX: 1703.10166

Cosmological collider physics

☐ Arkani-Hamed & Maldacena: 1503.08043
☐ Lee, Baumann, Pimentel: 1607.03735
☐ Wang, ZZX: 1910.12876

§1. Cosmic Inflation.

Two basic facts:

1: Our universe is expanding.

2: ······ is homogeneous & isotropic at large scales.

In GR: ① $$ds^2 = -dt^2 + a^2(t)\,\underset{\substack{\downarrow \\ \text{comoving coord.}}}{d\vec{x}^{\,2}}$$

(FRW metric)

$a(t)$: scale factor. $\dot{a} > 0$.

(spatial curvature ignored)

② $$T^{\mu}{}_{\nu} = \begin{pmatrix} -\rho(t) & & & \\ & p(t) & & \\ & & p(t) & \\ & & & p(t) \end{pmatrix}$$

3 variables, $a(t)$, $\rho(t)$, $p(t)$

3 eqs.

Eq. 1: Einstein eq. (00):

$$G_{\mu\nu} = 8\pi G T_{\mu\nu} \Rightarrow 3M_{Pl}^2\left(\frac{\dot{a}}{a}\right)^2 = \rho$$ ①

Friedmann's eq.

$$M_{Pl} = (8\pi G)^{-1/2} \simeq 2.4\times 10^{18}\ \text{GeV}.$$

$$\frac{\dot{a}}{a} = H$$ Hubble par.

Eq. 2: Continuity eq. $\nabla_\mu T^{\mu\nu} = 0$

$$\Rightarrow \dot{\rho} + 3H(p+\rho) = 0$$ ②

Eq. 3: $p = p(\rho)$ — Eq. of state.

③

$p = w\rho$. $a(t) \propto t^{\frac{2}{3(1+w)}}$

Cold matter : $w \simeq 0$ $\rightarrow a(t) \propto t^{2/3}$

radiation : $w \simeq \frac{1}{3}$ $\rightarrow a(t) \propto t^{1/2}$

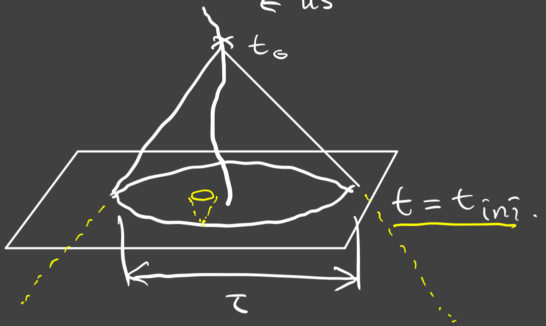
cosmo. const. $w \simeq -1$ $a(t) \propto e^{\underbrace{H}_{\text{const.}}t}$

$w = 0$: $\rho \propto a^{-3}$

$w = \frac{1}{3}$: $\rho \propto a^{-4}$

$w = -1$: $\rho \propto$ const.

Horizon problem.



$ds = 0 \Rightarrow dt = a\,dx$

$$\tau = \int_{t_{ini}}^{t_0} dx = \int_{t_{ini}}^{t_0} \frac{dt}{a}$$

$\Rightarrow \tau \propto t^{1/3}$ $(w=0)$

$\propto t^{1/2}$ $(w = \frac{1}{3})$

CMB : 0.4 myrs. $(w=0)$

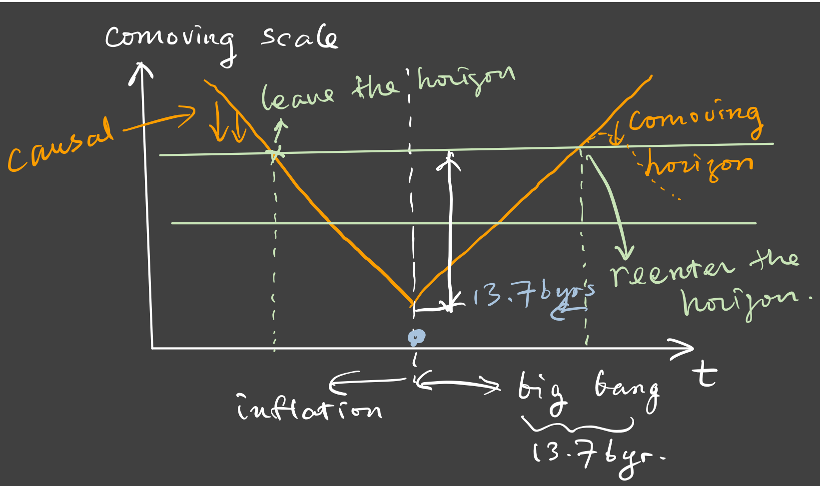
$$\Rightarrow \left(\frac{4 \times 10^5}{1.3 \times 10^{10}}\right)^{1/3} \simeq 0.03$$

Cosmic Inflation. (Solution)

(Guth. Linde. Starobinsky 1970s)

τ with $w = -1$

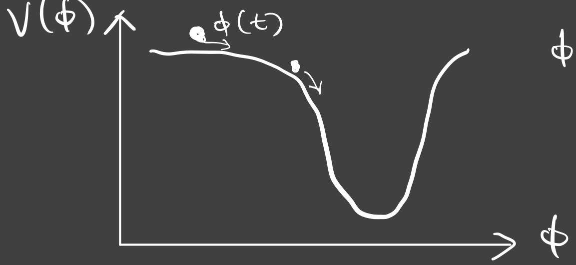
$\Rightarrow \tau \in (-\infty, 0)$



— Background dynamics.

cosmo. const. → ϕ (inflaton)

$$\mathcal{L} = -\frac{1}{2}(\partial_\mu \phi)^2 - V(\phi)$$



$$\phi = \phi(t)$$

$$\Box\phi - V'(\phi) = 0 \qquad (-+++)$$

$$\Rightarrow \ddot\phi + \underline{3H\dot\phi} + V'(\phi) = 0$$

Inflation $\Rightarrow V(\phi) >> \dot\phi^2$

$$\rho = \frac{1}{2}\dot\phi^2 + V(\phi) \;/\; p = \frac{1}{2}\dot\phi^2 - V(\phi)$$

$$w \simeq -1 \quad (\rho = \omega p)$$

$$\Rightarrow w = -\frac{1}{3} \Leftrightarrow \dot\phi^2 = V(\phi) :$$

inflation stops.

□ Slow condition :

$$\epsilon = \frac{\dot\phi^2}{2M_{Pl}^2 H^2} \qquad \underline{\epsilon << 1.}$$

$$\eta = \frac{\dot\epsilon}{H\epsilon} \qquad \underline{\eta << 1.}$$

$$* \; \epsilon = -\frac{\dot H}{H^2} \quad (\text{using eq. of motion})$$

"e-fold" $a(t_2) \to e\, a(t_1)$

$$e \simeq 2.7$$

How many e-folds do we need?
(to solve the horizon problem)

$$e^N = \frac{a_{today}}{a_{end}} = \underline{\underline{\frac{a_{today}}{a_{eq}}}}\ \frac{a_{eq}}{\underset{\substack{\downarrow \\ \text{reheating.}}}{a_{reh}}}\ \underline{\frac{a_{reh}}{a_{end}}}.$$

$$\frac{a_{today}}{a_{eq}} \sim e^7$$

$$\Rightarrow N = 7 + \frac{1}{4}\log\frac{\rho_{reh}}{\rho_{eq}} + \frac{1}{3}\log\frac{\rho_{end}}{\rho_{reh}}$$

$$(\rho \propto a^{-4}) \qquad (\rho \propto a^{-3})$$

$\rho_{end} \sim 10^{16}$ GeV $\Rightarrow$ $N \simeq 65$

$a_{end} \sim a_{reh}$

Lowest possible inflation scale

(BBN): 10 MeV $\Rightarrow$ $\underline{N \simeq 23.}$

Fluctuations.

$$S = \int d^4x\sqrt{-g}\left[\frac{M_{Pl}^2}{2}R - \frac{1}{2}(\partial\phi)^2 - V(\phi)\right]$$

$$\phi(t,\vec{x}) = \underbrace{\phi_0(t)}_{bg.} + \varphi(t,\vec{x})$$

$$\underline{\underline{ds^2}} = -N^2dt^2 + \underline{\underline{h_{ij}}}(dx^i + N^i dt) \times (dx^j + N^j dt)$$

N: lapse function (ADM)

N^i: shift vector.

$$\Rightarrow S = \frac{M_{Pl}^2}{2}\int dt\, d^3x \sqrt{h}\, N\Big[R^{(3)} + \frac{N^{-2}}{M_{Pl}^2}(E_{ij}E^{ij} - E^2) - \frac{2}{M_{Pl}^2}V - \frac{1}{M_{Pl}^2}h^{ij}(\partial_i\phi)(\partial_j\phi)\Big]$$

$E \equiv E_{ij}$

$$E_{ij} \equiv \frac{1}{2}(\dot{h}_{ij} - \nabla_i N_j - \nabla_j N_i)$$

$$K_{ij} = N^{-1} E_{ij} \qquad \phi = \phi_0 + \underline{\underline{\varphi}}$$

Introduce fluctuations:

$$N = 1 + \underline{\alpha}, \qquad N_i = \partial_i \underline{\psi} + \beta_i$$

$$h_{ij} = a^2 e^{2\underline{\zeta}} \left(\delta_{ij} + \underline{\gamma_{ij}} + \partial_i \kappa_j + \partial_j \kappa_i + \partial_i \partial_j \underline{\lambda} \right)$$

$$\partial^i \beta_i = \partial^i \kappa_i = 0. \qquad \partial^i \gamma_{ij} = 0, \quad \gamma^i{}_i = 0$$

Tensor (GW)

Remark: Modes of different spins do not mix at 2nd order in the action. But they do at higher orders.

Scalar perturbations

$$(\alpha, \psi, \zeta, \lambda, \varphi)$$

— Gauge transformation.

$$\delta g_{\mu\nu} = -\bar{g}_{\mu\lambda} \partial_\nu \epsilon^\lambda - \bar{g}_{\lambda\nu} \partial_\mu \epsilon^\lambda - \epsilon^\lambda \partial_\lambda \bar{g}_{\mu\nu}$$

$$\delta\phi = -\epsilon^\lambda \partial_\lambda \bar{\phi} \qquad (x^\mu \to x^\mu + \epsilon^\mu)$$

ϵ^λ: infinitesimal coord. transf.

$$\Rightarrow \epsilon^0 \equiv \chi \; ; \quad \epsilon^i = \partial^i \xi$$

(Ex.)

$$\Rightarrow \begin{cases} \delta\alpha = -\dot{\chi} \\ \delta\psi = \chi - \dot{\xi} \\ \delta\zeta = -H\chi \longleftarrow (\zeta\text{-gauge}) \\ \cancel{\delta\lambda} = -\dfrac{2}{a^2}\xi \\ \cancel{\delta\varphi} = -\dot{\phi}_0 \chi \longleftarrow \end{cases}$$

$$S_2 = M_{Pl}^2 \int dt\, d^3x \Big[-a^3(3-\epsilon)H^2\alpha^2$$

$$+ \left(-2aH\nabla^2\psi - 2a\nabla^2\zeta + 6a^3H\dot\zeta\right)\alpha$$

$$+ 2a(\nabla^2\psi)\dot\zeta - 3a^3\dot\zeta^2$$

$$- a\left(18a^2H\dot\zeta + \nabla^2\zeta\right)\zeta - 9a^3H^2(3-\epsilon)\zeta^2\Big]$$

$\epsilon \equiv \dot\phi_0^2/(2H^2)$, $\qquad \nabla^2 \equiv \delta^{ij}\partial_i\partial_j$

$\dfrac{\delta S_2}{\delta\alpha} = 0$, $\qquad \dfrac{\delta S_2}{\delta\psi} = 0$,

$$\Rightarrow \quad \alpha = \frac{\dot\zeta}{H}$$

$$\nabla^2\psi = -H^{-1}\nabla^2\zeta + a^2\epsilon\dot\zeta$$

$$\Rightarrow \quad S_2 = M_{Pl}^2 \int dt\, d^3x\, \epsilon \left[a^3\dot\zeta^2 - a(\partial_i\zeta)^2\right]$$

① $t \to \tau$; $\quad a\,d\tau = dt$

$$\Rightarrow \quad ds^2 = \underline{a^2(\tau)}\left[-d\tau^2 + d\vec{x}^2\right]$$

$$\Rightarrow \quad S_2 = \frac{1}{2}\int d\tau\, d^3x\, z^2\left[(\zeta')^2 - (\partial_i\zeta)^2\right]$$

$$\zeta' \equiv d\zeta/d\tau$$

$$z \equiv \sqrt{2\epsilon}\, M_{Pl}\, a$$

New variable (canonically normaliz

$u \equiv z\zeta$

$$\Rightarrow \quad S_2 = \frac{1}{2}\int d\tau\, d^3x \left[(u')^2 - (\partial_i u)^2 + \frac{z''}{z}u^2\right]$$

$$\Rightarrow \quad u'' - \partial_i\partial^i u - \frac{z''}{z}u = 0$$

$$u'' - \partial_i \partial^i u - \frac{z''}{z} u = 0$$

$$u(\tau, \vec{x}) = \int \frac{d^3\vec{k}}{(2\pi)^3} \left[\underset{\substack{\uparrow \\ \text{mode function.}}}{u_k(\tau)} e^{i\vec{k}\cdot\vec{x}} + c.c. \right]$$

$$\Rightarrow \quad u_k'' + \left(k^2 - \frac{z''}{z}\right) u_k = 0$$

Now consider a limit $\epsilon \to 0$.
(slow-roll).

$$a = e^{Ht} = -\frac{1}{H\tau} \Rightarrow \frac{z''}{z} = \frac{2}{\tau^2}$$

$$\Rightarrow u_k = \frac{1}{\sqrt{2k}} \left[c_1 \left(1 - \frac{i}{k\tau}\right) e^{-ik\tau} + c_2 \left(1 + \frac{i}{k\tau}\right) e^{+ik\tau} \right]$$

$$\zeta_k(\tau) = \frac{H}{2M_{Pl}\sqrt{\epsilon k^3}} \left[\boxed{c_1}\left(1 + ik\tau\right) e^{-ik\tau} + \boxed{c_2}\left(1 - ik\tau\right) e^{+ik\tau} \right]$$

Early time limit $\tau \to -\infty$. $\Rightarrow e^{\pm ik\tau}$

Late time limit $\tau \to 0$. $\Rightarrow \zeta \sim$ const.

Quantization of u.

$$u(\tau, \vec{x}) = \int \frac{d^3k}{(2\pi)^3} \left[u_k(\tau) e^{+i\vec{k}\cdot\vec{x}} a_{\vec{k}} + u_k^*(\tau) e^{-i\vec{k}\cdot\vec{x}} a_{\vec{k}}^\dagger \right]$$

momentum conjugate to u : $\frac{\partial \mathcal{L}}{\partial u'} = u'$

$$\begin{cases} [u(\tau, \vec{x}), u'(\tau, \vec{y})] = i\delta^{(3)}(\vec{x} - \vec{y}) \\ [a_{\vec{k}}, a_{\vec{p}}^\dagger] = (2\pi)^3 \delta^{(3)}(\vec{k} - \vec{p}) \end{cases}$$

$$\Rightarrow \underline{u_k(\tau)\, u_k'^*(\tau) - u_k^*(\tau)\, u_k'(\tau) = i} \Big\downarrow$$

Condition 2: $\underline{a_{\vec k}|0\rangle = 0}$ Wronskian

Question: $H|0\rangle = ?$

$$H = \frac{1}{2}\int d^3x \left[(u')^2 + (\partial_i u)^2 - \frac{z''}{z} u^2 \right]$$

$$\Rightarrow H|0\rangle = \frac{1}{2}\int \frac{d^3k}{(2\pi)^3}\Big[\underset{\overset{\parallel}{0}}{\boxed{\left((u_k'^*)^2 + \omega^2 (u_k^*)^2\right)}} \times a^\dagger_{-\vec k} a^\dagger_{\vec k}$$

$$+ \left(|u_k'|^2 + \omega^2 |u_k|^2 \right)(2\pi)^3 \delta^{(3)}(\vec 0) \Big] \underline{|0\rangle} \swarrow$$

$$\omega^2 = k^2 - \frac{z''}{z}.$$

Bunch-Davies vacuum.

$$\Rightarrow u_k' = \pm i \omega u_k \quad + \text{ Wronskian} = i$$

$$\Rightarrow |u_k|^2 = \cancel{\mp} \frac{1}{2\omega} \quad \Rightarrow u_k' = -i\omega u_k$$

$$* \quad \tau \to -\infty, \quad \frac{z''}{z} \to 0 \quad \Rightarrow \boxed{u_k' = -i k u_k}$$

$$\Rightarrow \zeta = \frac{H}{2M_{pl}\sqrt{\epsilon k^3}} (1 + ik\tau)\, e^{-ik\tau + \underline{i\theta}} \swarrow$$

stochastic quantum phase.

* Power spectrum (2point correlator)

$$\lim_{\tau \to \tau_f \simeq 0} \langle 0 | \zeta_{\vec k}(\tau)\, \zeta_{\vec q}(\tau) | 0 \rangle$$

$$= (2\pi)^3 \delta^{(3)}(\vec k + \vec q) \cdot \frac{2\pi^2}{k^3} \cdot \underbrace{P_\zeta(k)}_{\Downarrow}$$

Power spectrum.

$$\Rightarrow P_\zeta(k) = \frac{H^2}{8\pi^2 \epsilon M_{pl}^2} = \frac{H^4}{(2\pi)^2 \dot\phi_0^2}$$

P indep. of k

$\Rightarrow$ Scale invariance.

$$P_\zeta(k) \simeq 2 \times 10^{-9}$$

§2. QFT in dS background.

$$ds^2 = \frac{-d\tau^2 + d\vec{x}^2}{\tau^2} \qquad (H=1)$$

*. Isometry?

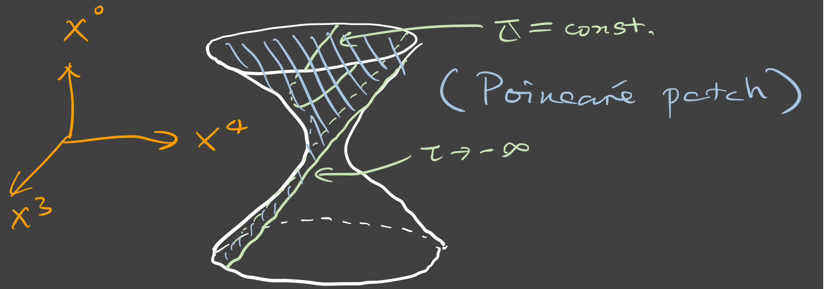
3d translation $P_i = \partial_i$

3d rotation: $J_i = \frac{1}{2}\epsilon_{ijk}(x_j\partial_k - x_k\partial_j)$

Dilation: $D = -\tau\partial_\tau - x^i\partial_i$

— Global dS: $\boxed{\eta_{MN}X^M X^N = 1}$

$M, N = 0, 1, \ldots, 4$



$$X^0 = -\frac{1-\tau^2+\vec{x}^2}{2\tau}$$

$$X^i = -\frac{x^i}{\tau} \qquad (i=1,2,3)$$

$$X^4 = -\frac{1+\tau^2-\vec{x}^2}{2\tau}$$

$SO(4,1)$,

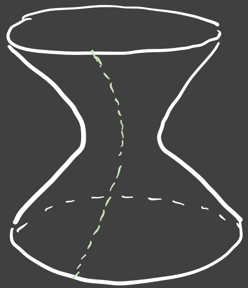
$$\mathcal{J}_{MN} = X_M\partial_N - X_N\partial_M$$

$$J_i = \frac{1}{2}\epsilon_{ijk}\mathcal{J}^{ik}$$

$$P_i = \mathcal{J}_{i0} - \mathcal{J}_{i4}$$

$$D = \mathcal{J}_{40}$$

$$K_i \equiv \mathcal{J}_{i0} + \mathcal{J}_{i4}$$



$$X^0 = \sinh\tau$$

$$X^i = \omega^i \cosh\tau$$

$(i=1,\dots,4)$

ω^i: (S^3)

$$\omega^1 = \cos\theta_1$$

$$\omega^2 = \sin\theta_1 \cos\theta_2$$

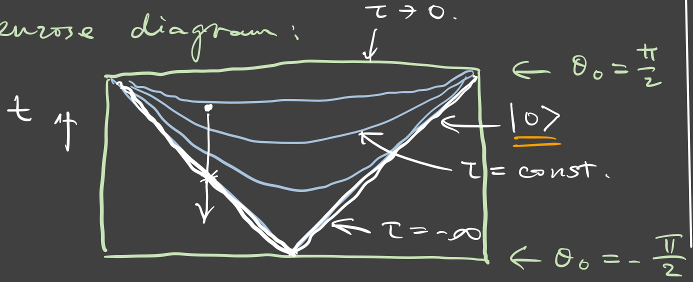
$$\omega^3 = \sin\theta_1 \sin\theta_2 \times \cos\theta_3 \quad \dots$$

$$\Rightarrow \quad ds^2 = -d\tau^2 + \cosh^2\tau \, d\Omega_3^2$$

$$\cosh\tau \equiv \frac{1}{\cos\theta_0}, \quad -\frac{\pi}{2} < \theta_0 < \frac{\pi}{2}$$

$$\Rightarrow \quad ds^2 = \frac{1}{\cos^2\theta_0}\left(-d\theta_0^2 + d\Omega_3^2\right)$$

⇒ Penrose diagram:



Inflation patch is geodesically incomplete [gr-qc/0110012 by Borde, Guth, Vilenkin]

Inflation patch breaks full dS.

(It breaks K_i)

- Quantum fields and States in dS.
- Case study.

(Massive scalar field σ)

$$(\Box - m^2)\sigma = 0.$$

$$\Rightarrow \sigma''(\tau, \vec{x}) - \frac{2}{\tau}\sigma'(\tau, \vec{x}) - \partial_i^2 \sigma(\tau, \vec{x}) + \frac{m^2}{H^2\tau^2}\sigma(\tau, \vec{x}) = 0$$

$$\Rightarrow \sigma_k(\tau) = \frac{\sqrt{\pi}}{2} e^{i\nu\pi/2} H(-\tau)^{3/2}$$

$$\times \underset{\uparrow}{H}{}^{(1)}_{\nu}(-k\tau)$$

Hänkel function of 1st kind.

$$\nu \equiv \sqrt{\frac{9}{4} - \left(\frac{m}{H}\right)^2} \qquad m > \frac{3}{2}H$$

$$m < \frac{3}{2}H.$$

Remarks:

Early time limit, $\sigma_k(\tau) \sim e^{-ik\tau}$

Late time limit $(k\tau \to 0)$.

$$\Rightarrow \lim_{\tau\to 0} \sigma_k(\tau) = -i\sqrt{\frac{2}{\pi k^3}}\, H$$

$$\times \left[e^{-i\nu\pi/2}\, \Gamma(-\nu) \left(-\frac{k\tau}{2}\right)^{3/2+\nu} + e^{+i\nu\pi/2}\, \Gamma(+\nu) \left(-\frac{k\tau}{2}\right)^{3/2-\nu} \right]$$

Consider $m \gg H$, (ν imaginary, $\nu \equiv i\tilde{\nu}$)

$m \gg H$, $\tilde{\nu} \simeq m/H$.

In terms of physical time t, $\left(a = e^{Ht} = -\frac{1}{H\tau}\right)$

$$\Rightarrow \sigma_k(t) = \frac{1}{\sqrt{2\tilde{\nu}H}} e^{-3Ht/2}$$

$$\times \left(e^{-i\tilde{\nu}Ht} b_{\vec{k}} + e^{+i\tilde{\nu}Ht} b^{\dagger}_{\vec{k}} \right)$$

Occupation number :

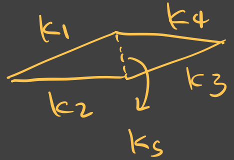
$$n_{\vec{k}} = \langle 0 | b_k^{\dagger} b_k | 0 \rangle$$

$$b_{\vec{k}} = \alpha_k a_{\bar{k}} + \underline{\underline{\beta_k}}\, a^{\dagger}_{\vec{k}}$$

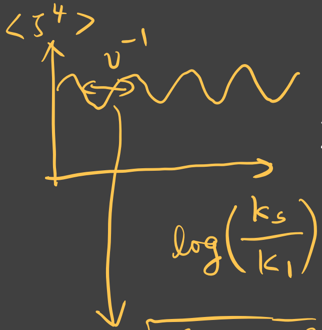
$$\Rightarrow n_{\vec{k}} = |\beta_k|^2$$

$$\Rightarrow |\beta_k|^2 = \frac{\tilde{\nu}^2}{2\pi} |\Gamma(-i\tilde{\nu})|^2 e^{-\tilde{\nu}\pi}$$

$\underline{m \gg H}$, $\tilde{\nu} = \frac{m}{H}$



$k_5 \ll k_1, k_2$

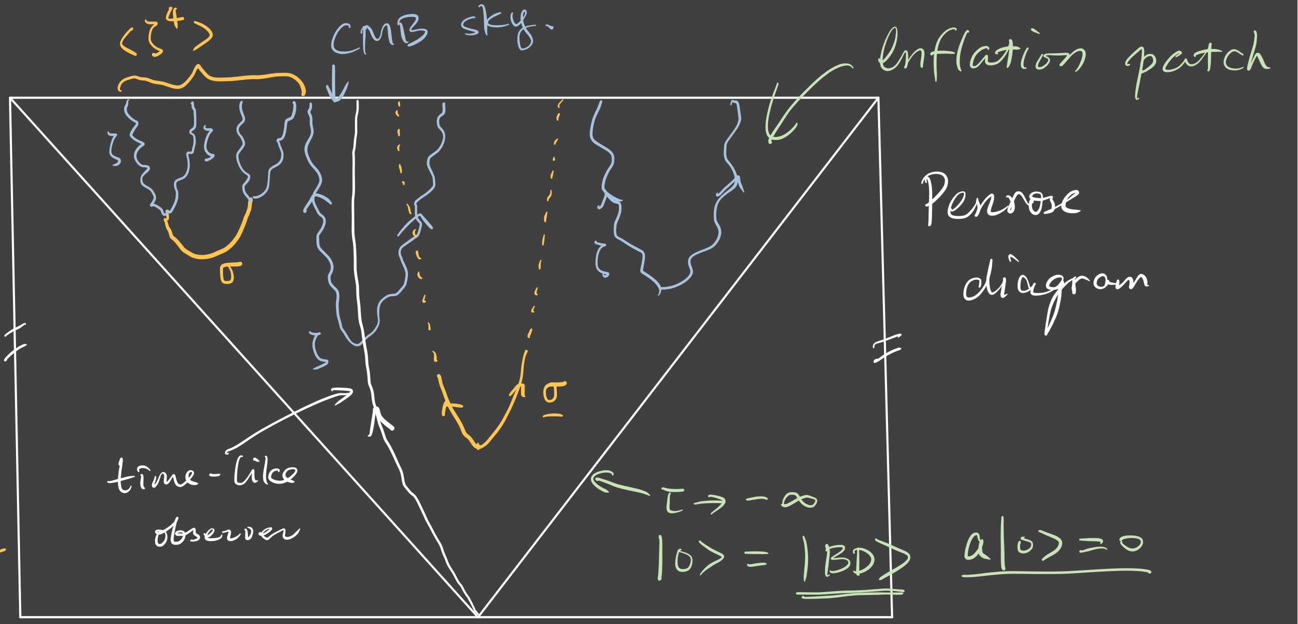


$$\nu = \sqrt{\frac{9}{4} - \left(\frac{m}{H}\right)^2}$$

$$m > \frac{3}{2}H$$

↓

Cosmological collider.



Interlude: Where are we?

SM: Higgs $\underline{125\,\text{GeV}}$.

$H \lesssim 10^{14}\,\text{GeV}$

$$|\beta_k|^2 = \frac{\tilde{\nu}}{2\pi}\left|\Gamma(-i\tilde{\nu})\right|^2 e^{-\tilde{\nu}\pi}$$

$m \gg H$, $\tilde{\nu} \simeq \frac{m}{H} \gg 1$,

$$\Rightarrow |\beta_k|^2 = \frac{\coth\tilde{\nu}\pi - 1}{2} \simeq \overset{\downarrow}{\boxed{e^{-2\pi m/H}}}$$

— Cosmic inflation can be used (viewed) as an engine for particle production. (for particle of mass $m \sim H$, $m \underset{<}{\gtrsim} H$)

$e^{-2\pi m/H}$: a first hint that $|BD\rangle$ of dS is a thermal state.

$$e^{-m/T} \Rightarrow T = \frac{H}{2\pi}$$

In general:

$$\lim_{\tau\to 0}\sigma(\tau,\vec{x}) = \sigma_+(\vec{x})(-\tau)^{\Delta_+} + \sigma_-(\vec{x})(-\tau)^{\Delta_-}$$

For scalar: $\Delta_\pm = \frac{3}{2} \pm \nu$, $\nu = \sqrt{\frac{9}{4} - \left(\frac{m}{H}\right)^2}$

Acting the 10 dS isometries $\sigma_\pm$:

$$P_i\,\sigma(\vec{x}) = \partial_i\,\sigma(\vec{x})$$

$$J_{ij}\,\sigma(\vec{x}) = \frac{1}{2}\epsilon_{ijk}\left(x_j\partial_k - x_k\partial_j\right)\sigma(\vec{x})$$

$$D\sigma(\vec{x}) = \left(-\Delta - x^i\partial_i\right)\sigma(\vec{x})$$

$$K_i\,\sigma(\vec{x}) = \left(-2\Delta x_i - 2x^i x^j\partial_j + \vec{x}^2\partial_i\right)\sigma(\vec{x})$$

dS isometry ⟺ Future infinity (3d slice) Conformal group

"dS/CFT"

– Classification of states.

[UIRs; Unitary irreducible rep. of $\underline{SO(4,1)}$]. (Weinberg QFT, V1)

$SO(3)$, $SO(1,1)$.

s : Spin. Δ : "conformal weight"

$\underline{s=0}$

principal series. $m > \frac{3}{2}H$

$$\Delta = \sqrt{\frac{9}{4} - \left(\frac{m}{H}\right)^2}$$

complementary series. $0 < m < \frac{3}{2}H$,

discrete series. ______

$s \neq 0$:

$$\Delta = \sqrt{\left(s-\frac{1}{2}\right)^2 - \left(\frac{m}{H}\right)^2}$$

P: $(m/H)^2 > \left(s-\frac{1}{2}\right)^2$

C: $\underline{s(s-1)} < \left(\frac{m}{H}\right)^2 < \left(s-\frac{1}{2}\right)^2$

↑ Higuchi bound.

d: ↘ $\left(\frac{m}{H}\right)^2 = s(s-1) - t(t-1)$

$t = 0, 1, \cdots, s-1$.

For $s=1$ or 2,

massless modes are in discrete series.

↓ ↓

"photon" graviton.

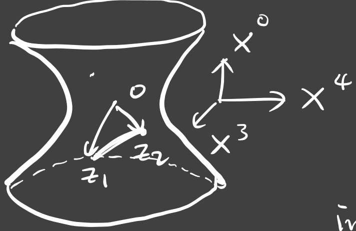
But, for $s=0$, $m=0$ is not in 1 particle state of dS.

"Allen": There exists no minimally coupled, massless scalar field 1PS in dS. [PRD $\underline{32}$ (1985) 3136]

Remarks: 1. Inflaton / $\underline{\zeta\text{-mode}}$ not affected. because slow-roll. $H \neq$ const.

2. Derivatively coupled scalar field not affected.

Imbedding distance.



$X^0, \cdots, X^4$.

$$\underline{Z = \eta^{MN} X_M X_N.}$$

$\downarrow$

imbedding distance.

In terms of inflation coordinates,

$$Z_{12} = \frac{\tau_1^2 + \tau_2^2 - |\vec{x}_1 - \vec{x}_2|^2}{2\tau_1\tau_2},$$

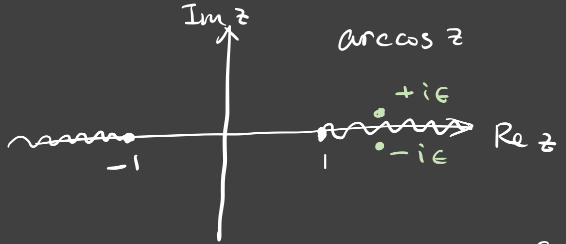
1: $(\tau_1, \vec{x}_1)$

2: $(\tau_2, \vec{x}_2)$.

$$\begin{cases} Z < 1 & \text{spacelike} \\ Z = 1 & \text{null} \\ Z > 1 & \underline{\underline{\text{timelike}}}. \end{cases}$$

Geodesic distance:

$$L_{12} = \underline{\arccos}\, Z_{12}$$

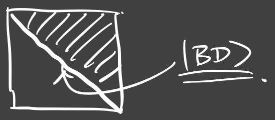


Time-ordering is needed for timelike separation:

$$\tilde{Z}(x_1, x_2) = Z(x_1, x_2) + \mathrm{sgn}(t_1 - t_2) \times i\epsilon.$$

$\underline{|BD\rangle}$ Vacuum state "BD" in dS.

is dS-invariant.



$\underline{a|BD\rangle = 0.}$

Proof: We show $\langle BD|\underline{\sigma(x_1)}\sigma(x_2)|BD\rangle$

$$\equiv \underline{G(x_1, x_2)}.$$

depends on x_1, x_2 through the combination $Z(x_1, x_2)$.

$$G(x_1,x_2) = \int \frac{d^3\vec{k}}{(2\pi)^3} e^{i\vec{k}\cdot(\vec{x}_1-\vec{x}_2)}$$

$$\times \langle 0|\sigma_{\vec{k}}(\tau_1)\sigma_{-\vec{k}}(\tau_2)|0\rangle$$

$$= \int \frac{dk\, k^2}{(2\pi)^2} d\cos\theta\, e^{ik x_{12}\cos\theta} \sigma_k(\tau_1)\sigma_k^*(\tau_2)$$

$$= \frac{e^{-\mathrm{Im}\,\nu\pi}}{8\pi x_{12}} H^2(\tau_1\tau_2)^{3/2}$$

$$\times \int dk\, k \sin(k x_{12}) H_\nu^{(1)}(-k\tau_1) H_{\nu^*}^{(2)}(-k\tau_2)$$

Consider a special case $\nu = \frac{1}{2}$,

$$G(x_1,x_2) = \frac{H^2\tau_1\tau_2}{4\pi^2(x_{12}^2-(\tau_1-\tau_2)^2)}$$

$$= \frac{H^2}{8\pi^2(1-Z_{12})} \quad ✓$$

For arbitrary ν, general dim D.

$$\underline{G(x_1,x_2)} = \frac{1}{(4\pi)^{D/2}} \frac{\Gamma(\mu_+)\Gamma(\mu_-)}{\Gamma(D/2)}$$

$$\times {}_2F_1\left(\mu_+, \mu_-; \frac{D}{2}; \frac{1+Z_{12}}{2}\right)$$

$$\mu_\pm = \frac{D-1}{2} \pm \sqrt{\frac{(D-1)^2}{4} - \left(\frac{m}{H}\right)^2}.$$

How to obtain $G(x_1, x_2)$ for general ν?

$$(\Box_{x_1} - m^2)\, \underline{G(x_1, x_2)} = 0.$$

$$\big\downarrow$$

$$\underline{G(Z)}.$$

$$\Rightarrow (1-Z^2)\, G''(Z) + D Z\, G'(Z) - \left(\frac{m}{H}\right)^2 G = 0$$

$$\Rightarrow G(Z) = \underline{\underline{c_1}} \times {}_2F_1\left(\mu_+, \mu_-; \frac{D}{2}; \frac{1+Z}{2}\right)$$

$$+ \underline{\underline{c_2}} \times {}_2F_1\left(\mu_+, \mu_-; \frac{D}{2}; \frac{1-Z}{2}\right)$$

${}_2F_1(a, b; c; z)$ has a pole at $z = 1$.

$\Rightarrow$ There exist a family of dS-inv. vacua, parametrized by $\alpha = \dfrac{c_2}{c_1}$

$\boxed{\text{“}\alpha\text{-vacua”.}}$ $\quad \underline{\alpha = 0 \Rightarrow \text{BD vacuum}}$

BD vacuum is thermal.

"Unruh detector". $\quad \underline{|E_i\rangle}$

$$m \sim c_1 \underline{\underline{a^\dagger}} + c_2 \underline{\underline{a}}$$

$$\mathcal{L} = \int d\tau\, g\, m(\tau)\, \underline{\underline{\sigma(x(\tau))}}.$$

$$\underset{\overset{\uparrow}{\sigma}}{|BD\rangle}\,\underset{\overset{\downarrow}{\text{observer}}}{|E_i\rangle} \longrightarrow \underset{\overset{\uparrow}{\sigma}}{|\beta\rangle}\,\underline{|E_j\rangle}$$

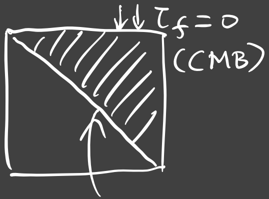
$$\Rightarrow \dot{P}(E_i \to E_j) = \underline{e^{-\beta(E_j - E_i)}\dot{P}(E_j - E_i)}$$

$$\Rightarrow \underline{\underline{N(E_i)}} \sim e^{-\beta E_i}$$

$$\beta = \frac{1}{T} = \frac{2\pi}{H}.$$

4. Path integral. [1703.10166]

Goal: $\langle \varphi(\tau_f,\vec{x}_1)\cdots\varphi(\tau_f,\vec{x}_n)\rangle$



$\uparrow$ $\tau_f \simeq 0.$ $\downarrow$

$\tau=-\infty$: $|BD\rangle$ $\underline{\langle BD|\varphi_1\cdots\varphi_n|BD\rangle}$

Idea:

$$\begin{tikzcd}
\langle BD|\cdots|BD\rangle \arrow[r] \arrow[d] & \sum\left(S\text{ matrix}\right)^2 \arrow[d] \\
\langle in|\cdots|in\rangle & \langle out|in\rangle
\end{tikzcd}$$

" in-in formalism "

At final slice $\tau_f=0$:

$$1=\sum_o |out\rangle\langle out|$$

$$\Rightarrow \langle BD|\varphi_1\cdots\varphi_n|BD\rangle$$

$$=\sum_{out}\underbrace{\langle BD|out\rangle}_{(S\text{ matrix})^\dagger}\underbrace{\langle out|\varphi_1\cdots\varphi_n|BD\rangle}_{S\text{ matrix}}$$

$$\langle out|\cdots|in\rangle=\int D\varphi\cdots e^{iS[\varphi]}$$

$$\boxed{\begin{aligned}&\Rightarrow \langle BD|\varphi_1\cdots\varphi_n|BD\rangle\\ &=\int D\varphi_- D\varphi_+\, e^{iS[\varphi_+]-iS[\varphi_-]}\,\varphi_{1+}\cdots\varphi_{n+}\\ &\quad\times\prod_{\vec{x}}\delta\left(\varphi_+(\tau_f,\vec{x})-\varphi_-(\tau_f,\vec{x})\right).\end{aligned}}$$

Schwinger–Keldysh path integral.

"Feynman rules" in SK formalism.

$$G_{++}(x_1,x_2) = \langle 0|T\{\varphi(\tau_1,\vec{x}_1)\,\varphi(\tau_2,\vec{x}_2)\}|0\rangle$$

$$G_{--} \cdots = \langle 0|\bar{T}\{\cdots\}|0\rangle$$

$$G_{+-} \cdots = \langle 0|\varphi(\tau_2,\vec{x}_2)\,\varphi(\tau_1,\vec{x}_1)|0\rangle$$

$$G_{-+} = \langle 0|\varphi(\tau_1,\vec{x}_1)\,\varphi(\tau_2,\vec{x}_2)|0\rangle$$

Bulk propagator

$\Rightarrow$ momentum space. (3d momentum)

$$G_{\underset{\overset{\swarrow}{\pm}}{\underline{ab}}}(\underset{\overset{\downarrow}{3d}}{k};\tau_1,\tau_2) = -i\int d^3\vec{x}\, e^{-i\vec{k}\cdot\vec{x}}\, G_{ab}(x_1,x_2)$$

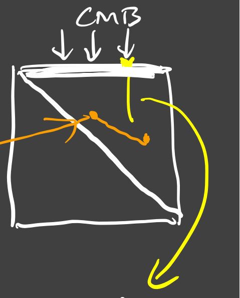
All four propagators in momentum space can reduced to a single object

$$\boxed{G_<(k;\tau_1,\tau_2) = u_k(\tau_1)\, u_k^*(\tau_2)}$$

$$G_{++} = G_>\,\theta(\tau_1-\tau_2) + G_<\,\theta(\tau_2-\tau_1)$$

$$(G_> = G_<^*)$$

. . . .



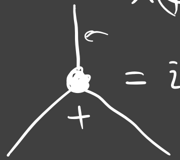
— Bulk-to-boundary propagator

$$G_\pm(k,\tau) = G_{\pm +}(k;\tau,\tau_f)$$

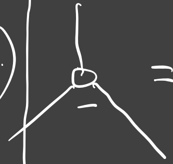
$= G_{++}$

$= G_{-+}$ . . .

$\lambda(\varphi')^3$

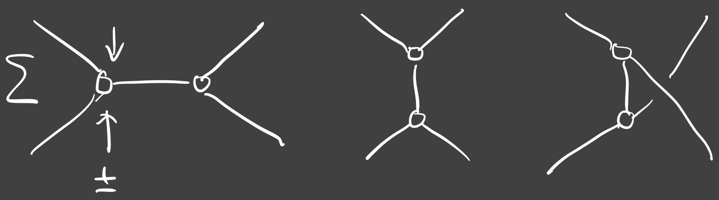


$= i\lambda(\partial_\tau \cdots)$



$= -i\lambda(\partial_\tau \cdots)$

$\langle \varphi\varphi\varphi\varphi \rangle$



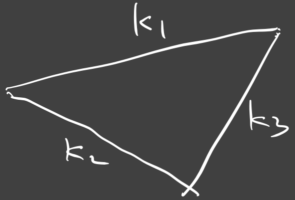
3-point correlator of ζ-mode in slow-roll inflation:

(Bispectrum). "4pT: trispectrum"

$$\boxed{P_\zeta \sim 2\times 10^{-9}}$$

$$\langle \zeta_{\vec{k}_1}\zeta_{\vec{k}_2}\zeta_{\vec{k}_3}\rangle' = \frac{(2\pi)^4 P_\zeta^2}{(k_1k_2k_3)^2} S(k_1,k_2,k_3)$$

$\downarrow$ Shape function.



$S(k_1/k_3,\ k_2/k_3)$

"Scale invariant and dimensionless"

$$S = \int d^4x\sqrt{-g}\left[\frac{M_p^2}{2}R - \frac{1}{2}(\partial\phi)^2 - V(\phi)\right]$$

$$S_3 = M_{pl}^2\int d\tau\, d^3\vec{x}\left\{a^2\epsilon^2\left[\boxed{\zeta(\zeta')^2} + \zeta(\partial_i\zeta)^2 - 2\zeta'\partial_i\zeta\,\partial_i^{-1}\zeta'\right] - \frac{1}{2}\partial_\tau\left(\epsilon\eta a^2\zeta^2\zeta'\right)\right\}$$

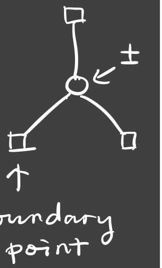
+ higher order in ϵ, η.

— Maldacena: astro-ph/0210603

Wang: 1303.1523.

$$M_{pl}^2\, \overset{\overset{1/(H\tau)^2}{\uparrow}}{\underline{a}}{}^2\, \epsilon^2\, \zeta\, (\zeta')^2 \qquad \zeta' = d\zeta/d\tau$$

$$\Rightarrow \langle \zeta_{\vec{k}_1} \zeta_{\vec{k}_2} \zeta_{\vec{k}_3} \rangle' \leftarrow \cancel{\delta(k_1+k_2+k_3)}$$



$$= 2i M_{pl}^2 \epsilon^2 \sum_{a=\pm} a \int_{-\infty}^{\tau_f} \frac{d\tau}{(H\tau)^2}$$

$$\times\ G_a(k_1, \tau)\, \partial_\tau G_a(k_2,\tau)\, \partial_\tau G_a(k_3,\tau)$$

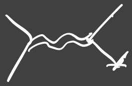
$$+\ 2 \text{ perms.}$$

$$= \frac{H^4}{4 M_{pl}^2 \epsilon k_1^3 k_2 k_3 k_t} + 2 \text{ perms.}$$

$$\Rightarrow S_{(1)} = \epsilon \left(\frac{k_1 k_2}{k_3 k_t} + 2 \text{ perms} \right)$$

$$k_t = |\vec{k}_1| + |\vec{k}_2| + |\vec{k}_3|$$

$$\Rightarrow S = \epsilon \left(\frac{k_1 k_2}{k_3 k_t} + 2 \text{ perms} \right) + \frac{\epsilon}{8} \left(\frac{k_1}{k_2} + 5 \text{ perms} \right) + \frac{\eta - \epsilon}{8} \left(\frac{k_1^2}{k_2 k_3} + 2 \text{perms} \right)$$



Remarks: 1. All terms are slow-roll suppressed.

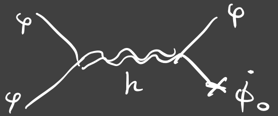
$$\boxed{\epsilon,\ \eta < 10^{-2}.}$$

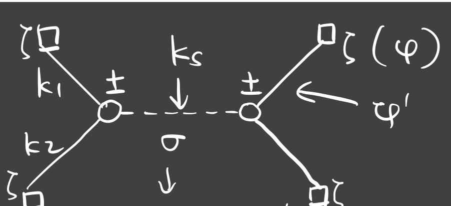
"consistency relation"

$\Delta(S) \sim O(0.1)$ [1210.7792] OPE.

2. $\langle \zeta^3 \rangle$ are dominated by graviton exchange.

(1811.00024)





massive scalar "m"

$$\begin{array}{c}\zeta \\ \downarrow \\ \varphi\end{array}$$

$$\zeta \Rightarrow -\frac{H}{\dot\phi_0}\varphi.$$

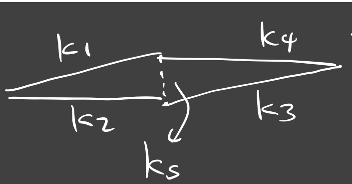
$$\frac{1}{\Lambda}\sqrt{-g}(\partial_\mu\phi)^2\sigma \Rightarrow \underline{\frac{1}{\Lambda}a^2(\varphi')^2\sigma}$$

$$\Rightarrow \langle \varphi_{\vec{k}_1}\cdots\varphi_{\vec{k}_4}\rangle'$$

$$= \frac{1}{\Lambda^2}\frac{1}{16k_1\cdots k_4}\sum_{a,b=\pm} ab\int_{-\infty}^{0} d\tau_1\int_{-\infty}^{0} d\tau_2$$

$$e^{iak_{12}\tau_1+ibk_{34}\tau_2}\underbrace{D_{ab}}_{\sigma\text{-propagator}}(k_s;\tau_1,\tau_2)$$

[1811.00024]
for full analytical result.



"collapse limit"

$$\underline{\underline{(m > \tfrac{3}{2}H)}}$$

$$k_s \ll k_1,\cdots,k_4$$

$$\lim_{\tau_1,\tau_2\to 0} D(k,\tau_1,\tau_2) = D_\ell + \underline{\underline{D_{n\ell}}}$$

$$D_{n\ell}(k;\tau_1,\tau_2) = \frac{H^2}{4\pi}(\tau_1\tau_2)^{3/2}$$

$$\times\left[\Gamma^2(-i\tilde\nu)(k^2\tau_1\tau_2)^{+i\tilde\nu} + (\tilde\nu\to-\tilde\nu)\right]$$

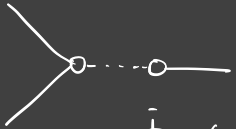
$$\tilde\nu = \sqrt{\left(\frac{m}{H}\right)^2 - \frac{9}{4}}$$

Cosmological collider signal.
↓

$$\boxed{\begin{aligned}&\Rightarrow \langle\varphi_1\cdots\varphi_4\rangle'_{n\ell}\\ &= A(m,\Lambda)\frac{1}{k_1k_2k_3k_4(k_{12}k_{34})^{5/2}}\\ &\quad\times \sin\left[\tilde\nu\,\underline{\underline{\log}}\frac{k_s^2}{k_{12}k_{34}} + \mathcal{O}(\tilde\nu)\right]\end{aligned}}$$

$$A(m,\lambda) \underset{m\gg H}{\simeq} \frac{\pi H^2}{8\Lambda^2}\left(\frac{m}{H}\right)^3 \underbrace{e^{-\pi m/H}}_{\text{Boltzmann suppression}}$$

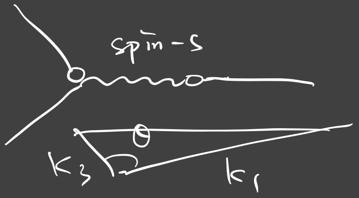
— Bispectrum.



$$\lim_{k_3/k_1\to 0} S(k_1,k_2,k_3) \sim \frac{\dot{\phi}_0}{\Lambda^2}\left(\frac{m}{H}\right)^{2/3}$$

$$\times\, e^{-\pi m/H}\left(\frac{k_1}{k_3}\right)^{-1/2}$$

$$\times\, \sin\left[\tilde{\nu}\log\frac{k_1}{k_3}+\varphi\right]$$

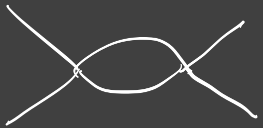


$$\propto P_s(\cos\theta)$$

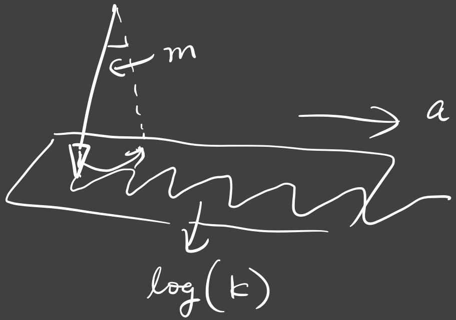
↓

"angular momentum conservation"

— Loop effect.



$$\sim \sin\left[2\tilde{\nu}\log\frac{k_s^2}{k_{12}k_{34}}+\varphi\right]$$



$$\longrightarrow a(t)\sim e^{Ht}$$

Interpretation of the oscillatory signal: Standard clock

(Chen, Namjoo, Wang 2014)

Infrared problem.

— $\lambda\phi^4$ with massless ϕ in dS.

$$\langle\phi^4\rangle \sim \mathrm{Im}\int_{-\infty}^{\tau_f} d\tau\, a^4(\tau)\left[(1+ik\tau)e^{-ik\tau}\right]^4$$

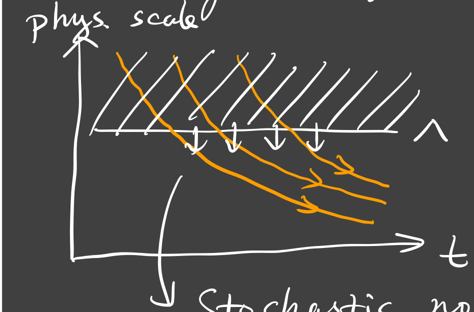
$$\sim \int \frac{d\tau}{\tau^4}\cdot\tau^3 \sim \log\tau_f \sim t_f$$

What happened to a self-int. massless scalar in dS?

— Starobinsky & Yokoyama astro-ph/9407016.

Question. What is a Wilsonian EFT in dS?

Choosing a physical cutoff Λ.



Stochastic noise of quantum origin.

$$\Rightarrow \phi_s(\tau,\vec{x}) = \int\frac{d^3k}{(2\pi)^3}\,\theta(k-\underset{\uparrow}{\epsilon}aH)$$

$$\times\left[\phi_k a_k e^{i\vec{k}\cdot\vec{x}} + c.c.\right]$$

$$\phi = \phi_L + \phi_s$$

$$\underline{\underline{\ddot{\phi}}} + 3H\dot{\phi} + V'_\phi = 0$$

$\underrightarrow{\phi = \phi_L + \phi_S}$, $\ddot{\phi} \simeq 0$

$$3H(\dot{\phi}_L + \dot{\phi}_S) + V'_\phi(\phi_L) = 0$$

↑ time derivative on θ

↓ leads to a noise term

$$\dot{\phi}_S \sim \theta(k - \epsilon a H)\phi_k$$

noise, $\langle f^2 \rangle \sim \frac{H^3}{4\pi^2}$

$$\Rightarrow \dot{\phi}_L + \frac{V'}{3H} = \underline{\underline{f(t, \vec{x})}}$$

$$\downarrow$$

$$\dot{\phi}_S \to \dot{\theta}$$

— Langevin equation.

$\Rightarrow$ Fokker-Planck:

$$\Rightarrow \underline{\underline{\dot{P}(\phi)}} = \frac{1}{3H}\frac{\partial}{\partial\phi}\left[V'(\phi)P(\phi)\right] + \langle f^2 \rangle \partial_\phi^2 P(\phi) = 0$$

$$\Rightarrow \langle\phi^2\rangle \sim \sqrt{\frac{3}{2\pi^2}}\,\frac{\Gamma(3/4)}{\Gamma(1/4)}\,\boxed{\frac{H^2}{\sqrt{\lambda}}}\,.$$

"nonperturbative result"

More recent development:

Baumgart & Sundrum, 1912.09502.

Gorbenko & Senatore: 1911.00022.

Rajaraman: 1008.1271.

Entanglement entropy of dS:

Pimentel & Maldacena